Adaptive Rational Interpolation: Restarting Methods for a Modified Rational Arnoldi Algorithm

Michalis Frangos and Imad M. Jaimoukha

***Abstract*— An algorithm which belongs to the class of Krylov projection methods for model reduction of linear systems is the rational Arnoldi algorithm. The resulting reduced models approximate the dynamics of the full order system at different interpolation points and a careful selection of the interpolation points can result in good approximations. However, the order of the reduced system can be relatively high while the error of approximation remains unsatisfactory. In this paper we develope numerically efficient restart schemes which improve further the approximation without increasing the order of the approximation.**

I. Introduction

Consider a linear time-invariant single-input single-output system described by the state-space equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \tag{1}$$

where $x(t) \in \mathbb{C}^n$ denotes the state vector and $u(t)$ and $y(t)$ the scalar input and output signals, respectively. The system matrix $A \in \mathbb{C}^{n\times n}$ is assumed to be large and sparse, and B, $C' \in \mathbb{C}^n$. These assumptions are met by large scale models in many applications. The transfer function for the system in (1) is denoted as $G(s) = C(sI_n - A)^{-1}B \overset{s}{=} (A,B,C,0)$ or

$$G(s) \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]. \tag{2}$$

To simplify subsequent analysis and design based on the large n'th order model in (1), the model reduction problem seeks an approximate m'th order model of the form

$$\dot{x}_m(t) = A_m x_m(t) + B_m u(t), \qquad y_m(t) = C_m x_m(t)$$

where $x_m(t) \in \mathbb{C}^m$, $A_m \in \mathbb{C}^{m\times m}$, B_m, $C_m' \in \mathbb{C}^m$ and $m < n$. The associated lower order transfer function is denoted by

$$G_m(s) \overset{s}{=} \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & 0 \end{array}\right].$$

Recently, the following issues, related to the rational Arnoldi algorithm, were discussed by the authors.

- Simple Arnoldi-like equations in the standard Arnoldi form have been derived in the rational case [4], [6].
- Error analysis for the rational case has been derived comparable to the standard algorithms, residual error expressions and forward error estimates were derived which are important for a better choice of the interpolation points [4], [6].

Michalis Frangos frangos@mit.edu and Imad M. Jaimoukha i.jaimouka@imperial.ac.uk, Electrical & Electronic Engineering Department, Imperial College London, Exhibition Road, London SW7 2AZ.

- A computationally efficient modified rational Arnoldi algorithm suitable for adaptive interpolation was developed and analysed in detail. In addition a set of Arnoldi-like equations for the algorithm was also derived [5].

Another problem for rational Krylov methods is summarized below:

- When the reduced system approximates a very large scale system, the order of the approximation might be too large while the error of the approximation is not satisfactory. In these cases a further improvement in the approximation without increasing the order of the approximation is desirable.

The contribution of this paper is in the development of numerically efficient adaptive methods for improving further the approximation of the reduced order systems obtained by the rational Arnoldi algorithms. The developement of the work presented in this paper is feasible due to the derivation of the Arnoldi-like equations [4], [5] which describe the rational Arnoldi algorithms. The considered methods are based on restart schemes, in the sense that the reduced order model is compressed and information based on new interpolation points is introduced by restarting the algorithms.

Some backround information is briefly discussed in Section II and in Section III an adaptive algorithm based on an efficient modified rational Arnoldi algorithm is presented. The proposed restart method is described and analysed in detail in Section IV and the moment matching properties of the reduced order approximation are given in Section V. Section VI gives some suggestions on the selection of the interpolation points. Numerical examples of the proposed method can be found in Section VII and finally, conclusions are drawn in Section VIII.

II. Krylov projections for model reduction

The system in (1) can be expanded by Taylor series around an interpolation point $s_0 \in \mathbb{C}$ as $G(s) = \mu_0 + \mu_1(s - s_0) + \mu_2(s - s_0)^2 + \cdots$ where the Taylor coefficients μ_i are known as the moments of the system around s_0 and are related to the transfer function of the system and its derivatives evaluated at s_0. The approximation problem by moment matching is to find a lower order system $G_m(s)$ (Padé-type model) with transfer function expanded as $G_m(s) = \hat{\mu}_0 + \hat{\mu}_1(s - s_0) + \hat{\mu}_2(s - s_0)^2 + \cdots$ such that $\mu_i = \hat{\mu}_i$, for $i = 0, 1, \ldots, m-1$. In the case where $s_0 = \infty$ the moments are called Markov parameters and are given by $\mu_i \triangleq CA^iB$. The moments around a finite interpolation point $s_0 \in \mathbb{C}$ are known as shifted moments and they are defined as $\mu_i \triangleq C(s_0I - A)^{-i}B$.

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A. Moment matching by rational interpolation

The moment matching problem in the projection concept was first treated in [21]. Projecting the states of the high order system with the projector $P_m \triangleq V_m V_m'$ where $V_m \in \mathbb{C}^{n\times m}$ is a unitary matrix, a reduced order model is obtained as

$$G_m(s) \overset{s}{=} \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} V_m' A V_m & V_m' B \\ \hline C V_m & 0 \end{array}\right]. \quad (3)$$

A careful selection of V_m as the basis of certain Krylov subspaces results in moment matching. For $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^n$, a Krylov subspace $\mathscr{K}_m(A,B,s)$ is defined as

$$\mathscr{K}_m(A,B,s) \triangleq \text{colsp}\left[(sI_n - A)^{-1}B, \dots, (sI_n - A)^{-m}B\right], \quad \text{if } s \neq \infty \text{ and}$$
$$\mathscr{K}_m(A,B,s) \triangleq \text{colsp}\left[B, AB, \dots, A^{m-1}B\right], \quad \text{if } s = \infty$$

where colsp denotes column span. If $m = 0$, $\mathscr{K}_m(A,B,s)$ is defined to be the empty set.

In [9] the author showed how to construct the required bases of Krylov subspaces by an algorithm based on the numerically efficient rational Krylov method developed in [18], such that the reduced order system, obtained by projection, matches the moments of the system around multiple interpolation points. We refer to this problem as the (multi-point) rational interpolation problem (e.g. [1], [7]–[9], [14]) and the solution of the problem is stated in Theorem 2.1 below.

THOEREM 2.1: [9] Let $\mathbf{S} = \{s_1, s_2, \dots, s_K\} \subset \mathbb{C}$ be a set of distinct interpolation points, with multiplicities $m_{s_1}, m_{s_2}, \dots, m_{s_K}$. Suppose that $V_m \in \mathbb{C}^{n\times m}$ satisfies

$$\text{colsp}(V_m) \supseteq \mathscr{K}_{m_{s_1}}(A,B,s_1) \cup \dots \cup \mathscr{K}_{m_{s_K}}(A,B,s_K)$$

where $\sum_{k=1}^{K} m_{s_k} = m$. Then the reduced order system matches the first m_{s_k} moments around s_k, assuming that in the case where $s_k \neq \infty$ the matrices $(s_k I_n - A)$ and $(s_k I_m - A_m)$ are invertible.

B. Modified rational Arnoldi algorithm

In this paper we use a modified rational Arnoldi algorithm based on [18] and given in Algorithm 1. A detailed description, a breakdown analysis and the moment matching properties of this algorithm can be found in [5]. For simplicity in this paper the vector e_j in line 7 of Algorithm 1 is considered to be a column $e_j = [0\ 0\ \dots 0\ 1]' \in \mathbb{R}^{j\times 1}$. Other choices of e_j have been discussed in [5]. The importance of this algorithm is that, given any set of interpolation points $\mathbf{S} = \{s_1, s_2 \dots, s_m\}$ where an interpolation point s_k can be included multiple times in any ordering, the algorithm computes efficiently a unitary basis V_m which satisfies the requirements in Theorem 2.1.

The next result is needed to derive Arnoldi-like equations for the rational modified Arnoldi algorithm and for preserving these equations for the restart arlgorithm given in Section IV.

LEMMA 2.1: Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times 1}$ and $m < n$. Let $\mathbf{S}_m = \{s_1, \dots, s_m\} \subset \mathbb{C}$ be given and assume that s_i is not an eigenvalue of A for all $s_i \in \mathbf{S}_m$. Define

$$S_1^n \triangleq \begin{cases} I_n, & \text{if } s_1 = \infty \\ (s_1 I_n - A)^{-1}, & \text{if } s_1 \neq \infty \end{cases}$$

and for $j = 2, \dots, m$

$$S_j^n \triangleq \begin{cases} A, & \text{if } s_j = \infty \\ (s_j I_n - A)^{-1}, & \text{if } s_j \neq \infty \end{cases}.$$

For $j = 1, \dots, m-1$ let $0 \neq e_j \in \mathbb{C}^{j\times 1}$ be given. Let $v_1, \dots, v_m \in \mathbb{C}^{n\times 1}$ be as generated by Algorithm 1 and let $V_j = [v_1, \cdots, v_j]$. Then, for $j = 1, \dots, m-1$,

$$\text{colsp}([S_1^n B\ ,\ S_{j+1}^n V_j]) \subset \text{colsp}(V_{j+1}). \quad (4)$$

Proof: The proof can be found in [5]. ■

Algorithm 1 Modified rational Arnoldi

1: Inputs: $A,\ B,\ \mathbf{S} = \{s_1, s_2, \dots, s_m\}$.
2: Initialise: $tolerance > 0$, $V_j = [\ ]$,
3: **if** $\{s_1 = \infty\}$, $\hat{v}_1 = B$ **else** $\hat{v}_1 = (s_1 I_n - A)^{-1}B$ **end**
4: **if** $\{\|\hat{v}_1\| < tolerance\}$, Stop **end**
5: $v_1 = \hat{v}_1/\|\hat{v}_1\|$, $V_m = v_1$
6: **for** $\{j = 1 \to m-1\}$
7: $\quad \hat{v}_{j+1} = \begin{cases} AV_j e_j, & \text{if } s_{j+1} = \infty \\ (s_{j+1} I_n - A)^{-1} V_j e_j, & \text{if } s_{j+1} \neq \infty \end{cases}$
8: $\quad \hat{v}_{j+1} = \hat{v}_{j+1} - V_j V_j' \hat{v}_{j+1}$
9: $\quad$ **if** $\{\ \|\hat{v}_{j+1}\| < tolerance\}$, Stop, **end**
10: $\quad v_{j+1} = \hat{v}_{j+1}/\|\hat{v}_{j+1}\|$, $V_{j+1} = [\ V_j \quad v_{j+1}\ ]$
11: **end**
12: Output : V_{j+1}

The next result is also needed to prove the moment matching properties of the reduced model, using the modified Arnoldi algorithm. It is also used later in Section IV to prove the preservation of the results of Lemma 2.1 after restarting the algorithm.

LEMMA 2.2: Suppose that $\mathbf{S}_r = \{s_{i_1}, s_{i_2}, \dots, s_{i_r}\} \subseteq \mathbf{S}_m$, where $1 \le i_1 < \dots < i_r \le m$ and $1 \le r \le m$ and let the basis V_m constructed by the modified rational Arnoldi Algorithm as described in Lemma 2.1. Then,

$$(S_{i_r}^n \dots S_{i_1}^n)B \in \text{colsp}(V_m). \quad (5)$$

where

$$S_{i_1}^n \triangleq \begin{cases} I_n, & \text{if } s_{i_1} = \infty \\ (s_{i_1} I_n - A)^{-1}, & \text{if } s_{i_1} \neq \infty \end{cases}$$

and for $j = 2, \dots, r$

$$S_{i_j}^n \triangleq \begin{cases} A, & \text{if } s_{i_j} = \infty \\ (s_{i_j} I_n - A)^{-1}, & \text{if } s_{i_j} \neq \infty \end{cases}.$$

Proof: See the proof for Corollary 3.1 in [5]. ■

Corollary 2.1 below, establishes the moment matching properties of the reduced order model obtained by projection with the basis V_m constructed by the modified Arnoldi algorithm.

COROLLARY 2.1: Let V_m be the basis constructed by the modified rational Arnoldi procedure given in Algorithm 1 and suppose that $q \in \mathbf{S}_m$ has multiplicity p. Then

$$\mathscr{K}_p(A,B,q) \subseteq \text{colsp}(V_m).$$

and the reduced order system defined in (3) matches the first p moments of the original system around the interpolation point $q \in \mathbf{S}_m$.

Proof: This is a direct result from Lemma 2.2 and Theorem 2.1. ■

III. Adaptive rational Arnoldi algorithm and Arnoldi-like equations

A simpler case of the rational interpolation problem is solved by the standard Arnoldi process [2] which is a process that constructs a unitary basis $V_{m+1} \in \mathbb{C}^{n\times m+1}$ such that $\mathscr{K}_{m+1}(A,B,\infty) \subseteq \text{colsp}(V_{m+1})$. A set of equations known as the Arnoldi equations [11]–[13] are satisfied on every iteration of the standard Arnoldi algorithm; these equations are useful for error analysis, for deriving residual error expressions and stoping criteria for the algorithms, for perturbation analysis and restarts [10], [13], [20].

The next corollary how to derive with minimum additional effort, the Arnoldi-like equations in the rational case when the basis V_m is constructed by Algorithm 1, [4], [6].

Corollary 3.1: Let all variables be as given in Lemma 2.1. Suppose $v_{m+1} \in \mathbb{C}^n$ is defined such that, with $V_{m+1} \triangleq \begin{bmatrix} V_m & v_{m+1} \end{bmatrix}$,

$$\text{colsp}([V_m \quad AV_m e_m]) = \text{colsp}(V_{m+1}), \quad V'_{m+1}V_{m+1} = I_{m+1}.$$

$$\text{Define} \quad \begin{bmatrix} A_m & B_m \\ C_{V_m} & b_m \end{bmatrix} \triangleq \begin{bmatrix} V'_m A V_m & V'_m B \\ v'_{m+1} A V_m & v'_{m+1} B \end{bmatrix},$$

then the following Arnoldi-like equations are satisfied

$$AV_m = V_m A_m + v_{m+1} C_{V_m}, \tag{6}$$

$$B = V_m B_m + v_{m+1} b_m. \tag{7}$$

Furthermore, $b_m = 0$ if $\infty \in \mathbf{S}$ and the Arnoldi-like equations are in the same form as in the standard Arnoldi case.

Proof: It is easy to see that the construction of v_{m+1} is equivalent to running Algorithm 1 for one extra interpolation point at $s_{m+1} = \infty$, so that $S^n_{m+1} = A$. Thus a direct application of Lemma 2.1 gives (6).

To prove (7) we proceed as follows. If $s_1 = \infty$, so that $S^n_1 = I_n$, (4) gives $B \in \text{colsp}(V_{m+1})$ and the result is proved. If $s_1 \neq \infty$, so that $S^n_1 = (s_1 I_n - A)^{-1}$, then (4) gives $B = (s_1 I_n - A)V_{m+1}X_m$ for some X_m and so $B \in \text{colsp}(V_{m+1})$ from (6), which proves (7).

To prove the final part, suppose that $\infty \in \mathbf{S}_m$. Then Corollary 2.1 implies that $K_p(A,B,\infty) \subseteq \text{colsp}(V_m)$ for some $p \geq 1$. In particular, this implies that $B \in \text{colsp}(V_m)$ and so $b_m = v'_{m+1}B = 0$ and the result is proved. ■

In Krylov-subspace projection methods the residual error is defined as $R_B(s) \triangleq B - (sI - A)V_m X_m(s)$, where $X_m(s)$ is the solution to the system $(sI_m - A_m)X_m(s) = B_m$ and satisfies the Petrov-Galerkin conditions $R_B(s) \perp \text{colsp}(V_m)$, i.e., $V'_m R_B(s) = 0$. By substitution of the Arnoldi-like equations in the residual error expression given above, the residual error can be simplified as,

$$R_B(s) = v_{m+1}\tilde{R}_B(s), \quad \tilde{R}_B(s) \stackrel{s}{\triangleq} (A_m, B_m, C_{V_m}, b_m), \tag{8}$$

which involves reduced order terms only. The simple residual error expression in (8) can be used for an adaptive method for interpolation point selection; on every iteration of the modified Arnoldi algorithm the next interpolation point is choosen to be the frequency $j\omega_o$ at which $|\tilde{R}_B(j\omega_o)|$ is maximum. A pseudo-code for the adaptive Arnoldi algorithm [4] is given in Algorithm 2.

Algorithm 2 Adaptive rational Arnoldi algorithm

```
1: Inputs : A, B, m
2: Initialise: s_1, S_1 = {s_1}, V_j = [ ] and j = 0
3: while {Number of columns in V_j < m − 1},
4:     Compute v_{j+1} wrt s_{j+1}, V_{j+1} = [V_j  v_{j+1}], j = j+1.
5:     Generate the Arnoldi equations as in Corollary 3.1.
6:     Compute R̃_B and jω_o which maximizes |R̃_B(jω_o)|
7:     Set s_{j+1} = jω_o and S_{j+1} = {S_j, s_{j+1}}
8: end
9: Outputs: A_j = V_j'AV_j, B_j = V_j'B, S_j = {s_1,...,s_j}
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IV. Restarting the rational Arnoldi algorithm

In the case of the standard Arnoldi methods the reduced order models can be improved further by restarting the algorithms to filter out unwanted information, while keeping the order of the approximation low [16], [17]. In polynomial restarting methods after m steps of the Arnoldi algorithm, the algorithm recomputes a new basis $\tilde{V}_m$ such that $\text{colsp}(\tilde{V}_m) = \mathscr{K}_m(A, \psi(A)v_1)$ where $\psi(A)$ is a polynomial filter which removes the unwanted components from the starting vector. The computation of the starting vectors through matrix-vector computations yields to explicit restarting schemes [19]. A more efficient and more numerically stable way is offered by the approaches based on applying the polynomial filters implicitly by the use of orthogonal transformations. More on implicit restarts can be found for example in [10], [13], [20].

In this paper an alternative method for restarting the Algorithm is proposed so that no moment matching properties are lost. The implementation for the adaptive restart schemes presented, is based on efficient projections to replace moments around some interpolation points such that the overall approximation error is reduced further. The performance of the restart schemes depends on which moments are discarded and which moments are added. The restart methods in this paper are feasible due to the preservation of the Arnoldi-like equations, which in turn follows from Lemma 2.1.

The method consists of the following steps:

1) Construct the unitary basis $V_m \in \mathbb{C}^{n\times m}$ as described in Lemma 2.1 by the modified rational Arnoldi algorithm with inputs $(A, B, \mathbf{S}_m)$ and compute the reduced order system

$$G_m(s) \stackrel{s}{\triangleq} \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & 0 \end{array}\right]. \tag{9}$$

According to Corollary 2.1, G_m matches the moments of G at the interpolation points in $\mathbf{S}_m$.

2) Let $\mathbf{S}_r = \{s_{i_1}, s_{i_2}, \ldots, s_{i_r}\}$ such that

$$\mathbf{S}_r \subset \mathbf{S}_m,$$

where $1 \le i_1 < \ldots < i_r \le m$ and $1 \le r < m$. Then running the modified Arnoldi algorithm with inputs $(A_m, B_m, \mathbf{S}_r)$ we construct a unitary basis $V_{mr} \in \mathbb{C}^{m \times r}$.
According to Lemma 2.1 for $j = 1, \ldots, r-1$,

$$\text{colsp}([S^m_{i_1} B_m \ , \ S^m_{i_{j+1}} V^j_{mr}]) \subset \text{colsp}(V^{j+1}_{mr}),$$

where V^j_{mr} denotes the first j columns of V_{mr},

$$S^m_{i_1} \triangleq \begin{cases} I_m, & \text{if } s_{i_1} = \infty \\ (s_{i_1} I_m - A_m)^{-1}, & \text{if } s_{i_1} \neq \infty \end{cases}$$

and for $j = 2, \ldots, r$

$$S^m_{i_j} \triangleq \begin{cases} A_m, & \text{if } s_{i_j} = \infty \\ (s_{i_j} I_m - A_m)^{-1}, & \text{if } s_{i_j} \neq \infty \end{cases}$$

assuming $(s_{i_j} I_m - A_m)$ is non-singular, for any $s_{i_j} \in \mathbf{S}_r$.
According to Corollary 2.1 the system G_{mr} defined by

$$G_{mr}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_{mr} & B_{mr} \\ \hline C_{mr} & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} V'_{mr} A_m V_{mr} & V'_{mr} B_m \\ \hline C_m V_{mr} & 0 \end{array}\right] \quad (10)$$

matches the moments of G_m at the interpolation points in $\mathbf{S}_r$.
Remark 1: Since G_m matches the moments of G at $\mathbf{S}_r$ it follows that G_{mr} also matches the moments of G at the interpolation points in $\mathbf{S}_r$.
Remark 2: The construction of the basis V_{mr} is computationally inexpensive since the reduction method is applied on a reduced order system.

3) Next construct the basis $V_r \in \mathbb{C}^{n \times r}$ as,

$$V_r \triangleq V_m V_{mr},$$

and define the system G_r as

$$G_r(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} V'_r A V_r & V'_r B \\ \hline C V_r & 0 \end{array}\right]. \quad (11)$$

The next result proves that the equations of the modified rational Arnoldi algorithm presented in Lemma 2.1 are preserved after the restarts. This is only possible if $\mathbf{S}_r$ is a subset of $\mathbf{S}_m$ as determined in step 2).
LEMMA 4.1: Let the basis $V_r = V_m V_{mr}$ where V_m and V_{mr} are constructed by the modified Arnoldi algorithm as described in steps 1) and 2) respectively. Then, for $j = 1, \ldots, r-1$,

$$\text{colsp}([S^n_{i_1} B \ , \ S^n_{i_{j+1}} V^j_r]) \subset \text{colsp}(V^{j+1}_r) \quad (12)$$

where V^j_r denotes the first j columns of V_r.
Proof: Suppose that $\mathbf{S}_p = \{s_{k_1}, s_{k_2}, \ldots, s_{k_p}\}$ where $\mathbf{S}_p$ is any subset of $\mathbf{S}_r$ and $i_1 \le k_1 < \cdots < k_p < i_r$. From the results in Lemma 2.2 we have that, for $\mathbf{S}_1 = s_{k_1}$,

$$S^n_{k_1} B \in \text{colsp}(V_m). \quad (13)$$

Next we define a projector $P_{\mathscr{V}_m}$ onto colsp(V_m) as

$$P_{\mathscr{V}_m} \triangleq \begin{cases} V_m S^m_{k_j} V'_m (S^n_{k_j})^{-1}, & \text{if } s_{k_j} \neq \infty, \\ V_m V'_m, & \text{if } s_{k_j} = \infty. \end{cases}$$

Now if $v \in$ colsp(V_m) then projecting v by $P_{\mathscr{V}_m}$ gives itself, i.e., $P_{\mathscr{V}_m} v = v$. Therefore projecting the column in (13) with $P_{\mathscr{V}_m}$ for $j = 1$ we obtain,

$$S^n_{k_1} B = V_m S^m_{k_1} B_m. \quad (14)$$

Due to the construction of the basis V_{mr}, following similar steps as above and noting that $A_{mr} = V'_{mr} A_m V_{mr} = A_r$ and $B_{mr} = V'_{mr} B_m = B_r$ we obtain that

$$S^m_{k_1} B_m = V_{mr} S^r_{k_1} B_r, \quad (15)$$

where

$$S^r_{k_1} \triangleq \begin{cases} I_r, & \text{if } s_{k_1} = \infty, \\ (s_{k_1} I_r - A_r)^{-1}, & \text{if } s_{k_1} \neq \infty, \end{cases}$$

assuming $(s_{k_1} I_r - A_r)$ is non-singular. Substituting equation (15) into (14) gives

$$S^n_{k_1} B = V_r S^r_{k_1} B_r. \quad (16)$$

Next assume that

$$(S^n_{k_p} \ldots S^n_{k_1}) B = V_m (S^m_{k_p} \ldots S^m_{k_1}) B_m. \quad (17)$$

The assumption in (17) is proved for $\mathbf{S}_1 = s_{k_1}$ in (14). We proceed by induction to prove that the assumption is also true for $\mathbf{S}_p \cup s_{k_{p+1}}$, for any $s_{k_{p+1}} \in \mathbf{S}_r$ where $k_p < k_{p+1} \le r$.
Multiplying (17) from the left by $S^n_{k_{p+1}}$ we obtain,

$$(S^n_{k_{p+1}} S^n_{k_p} \ldots S^n_{k_1}) B = S^n_{k_{p+1}} V_m (S^m_{k_p} \ldots S^m_{k_1}) B_m. \quad (18)$$

From Lemma 2.2 we know that $(S^n_{k_{p+1}} S^n_{k_p} \ldots S^n_{k_1}) B \in$ colsp(V_m) therefore projecting (18) with $P_{\mathscr{V}_m}$ for $j = p+1$ gives,

$$(S^n_{k_{p+1}} S^n_{k_p} \ldots S^n_{k_1}) B = V_m (S^m_{k_{p+1}} \ldots S^m_{k_1}) B_m, \quad (19)$$

which proves that the assumption in (17) can be extended for any subset in S_r. Following similar steps as above, due to the construction of the basis V_{mr} the following relation can be verified,

$$(S^m_{k_{p+1}} \ldots S^m_{k_1}) B_m = V_{mr} (S^r_{k_{p+1}} \ldots S^r_{k_1}) B_r. \quad (20)$$

Finally substituting (20) in (19) results in

$$(S^n_{k_{p+1}} \ldots S^n_{k_1}) B = V_r (S^r_{k_{p+1}} \ldots S^r_{k_1}) B_r. \quad (21)$$

where $S^r_{k_1}$ is as previously defined and for $j = 2, \ldots, r$

$$S^r_{k_j} \triangleq \begin{cases} A_r, & \text{if } s_{k_j} = \infty \\ (s_{k_j} I_r - A_r)^{-1}, & \text{if } s_{k_j} \neq \infty \end{cases}$$

assuming $(s_{k_j} I_r - A_r)$ is non-singular.
Having proved that the relation in (21) holds for any subset in $\mathbf{S}_r$, this implies that the basis V_r satisfies the results of Lemma 2.2. This in turn implies that the basis V_r is as if constructed by the modified rational Arnoldi algorithm with inputs (A, B, S_r) and that equation (12) in Lemma 4.1 holds. ■

4) In the previous steps we have seen how to discard efficiently some information from the basis $V_m \in \mathbb{C}^{n \times m}$ related to some interpolation points to obtain the basis

$V_r \in \mathbb{C}^{n\times r}$. Most importantly we have seen that the results of Lemma 2.1 are preserved for the basis V_r at the set of interpolation points $\mathbf{S}_r \subset \mathbf{S}_m$. This implies that the Arnoldi-like equations can be derived as described in Corollary 3.1 and updated residual errors expressions can also be derived in the form of the residual expressions given in (8).
Therefore the modified rational Arnoldi process can be restarted by Algorithm 2 with the initialization $V_j = V_r$ and $\mathbf{S}_m = \mathbf{S}_r$. The whole restart process can be repeated until the desired error of approximation is reached.

V. Moment Matching Properties after restarts

After a number of restarts a new basis $\tilde{V}_m \in \mathbb{C}^{n\times m}$ and an updated set of interpolation points $\tilde{\mathbf{S}}_m = \{\tilde{s}_1 \ldots \tilde{s}_m\}$ are obtained. The reduced order of approximation is obtained by orthogonal projection as,

$$G_m(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} \tilde{V}_m' A \tilde{V}_m & \tilde{V}_m' B \\ \hline C\tilde{V}_m & 0 \end{array}\right]. \tag{22}$$

At the end of the restart process the results in Lemma 2.1 are preserved as proved in Lemma 4.1. Therefore according to Corollary 2.1 the model in (22) matches m moments of the original system at the interpolation points in $\tilde{\mathbf{S}}_m$.

The accuracy of the approximation after the restarts, depends on the selection of the new interpolation points in $\tilde{\mathbf{S}}_m$. In this paper on every iteration the next interpolation point was computed based on the norm of the residual error expression in (8). More accurate error estimates have been derived in [4] which can be used instead. On step 3) of the suggested method for restarting the modified Arnoldi Algorithm, it is required that, $\mathbf{S}_r$ should be a subset of $\mathbf{S}_m$. Obviously some choices of the subset $\mathbf{S}_r$ will result in better approximations that other subsets. This issue is addressed in the following section where a number of methods for choosing $\mathbf{S}_r$ are suggested.

VI. Choice of the subset of interpolation points

In this section a number of methods are suggested for choosing the subset $\mathbf{S}_r$ from $\mathbf{S}_m$. Different selections provide different results for the restart methods presented; a description of the methods is given below.

- Simple Selection Method (SSM): Let $\mathbf{S}_m$ be the set of interpolation points obtained by the adaptive rational Arnoldi algorithm given in Algorithm 2, prior to any restarts. On every iteration the algorithm constructs an expression for the residual error, based on which the next interpolation point in $\mathbf{S}_m$ is computed. The order of the residual expression and therefore its accuracy is increased on every iteration. This implies that as the order of the approximation increases the interpolation points computation tends to be more accurate.
  Therefore in this method, we suggest discarding the interpolation points computed at the earlier iterations,
  $$\mathbf{S}_r = \mathbf{S}_m \setminus \{s_1, \ldots, s_{m-r}\}.$$
- Optimised Selection (OSM): This is a more systematic way for selecting $\mathbf{S}_r$, though more computationally expensive. The OSM method is based on the idea of reducing the basis V_m to V_r incrementaly as follows.
  Start by choosing an interpolation point $s^* \in \mathbf{S}_m$ such that $\|G_m - G^*_{m-1}\|_\infty$ is minimized, where G^*_{m-1} is the interpolation of G_m at
  $$\mathbf{S}_{m-1} = \mathbf{S}_m \setminus s^*.$$
  The process can be repeated until $m-r$ interpolation points are discarded from $\mathbf{S}_m$.
- Efficient Optimised Selection Method (EOSM): The EOSM method is another approach for choosing a set $\mathbf{S}_r$ from $\mathbf{S}_m$ in a more efficient way than OSM. In this approach for each $s_i \in \mathbf{S}_m$ we evaluate
  $$e_i = \|G_m(s_i) - \hat{G}_r(s_i)\| / \|G_m(s_i)\|,$$
  where $\hat{G}_r$ is an approximation of G_m by a conventional method (e.g. balanced truncation). This is computationaly acceptable since the conventional reduction method is applied on a low order system. Then we choose the set $\mathbf{S}_r$ to contain the interpolation points corresponding to the smallest e_i's. The idea is that if $e_i = 0$ then $G(s_i) = G_m(s_i) = \hat{G}_r(s_i)$ and so we should keep s_i.

The restart process for the adaptive modified rational Arnoldi algorithm described in the steps above, is implemented below in Algorithm 3.

Algorithm 3 Adaptive modified rational Arnoldi with restarts

Inputs: $A, B, \mathbf{S}_m$, K, r, $\varepsilon > 0$ (tolerance);
$[V_m]$ = arnoldi_modified($[A, B, \mathbf{S}_m]$);
for $\{i = 1 \to K\}$
  Choose $\mathbf{S}_r \subset \mathbf{S}_m$; % Discussed in Section VI.
  $A_m = V_m' A V_m$; $B_m = V_m' B$;
  $[V_{mr}]$ = arnoldi_modified($[A_m, B_m, \mathbf{S}_r]$);
  $V_r = V_m V_{mr}$; $\tilde{V}_m^j = V_r$; $\tilde{\mathbf{S}}_j = \mathbf{S}_r$;
  for $\{j = r \to m-1\}$ % restart block
    Generate the Arnoldi equations as in Corollary 3.1.
    Compute $\tilde{R}_B$ and $j\omega_o$ which maximizes $|\tilde{R}_B(j\omega_o)|$
    Set $s_{j+1} = j\omega_o$ and update $\tilde{\mathbf{S}}_{j+1} = \{\tilde{\mathbf{S}}_j, s_{j+1}\}$;
    Set the vector $e_j \in \mathbb{C}^{j\times 1}$ as defined in Lemma 2.1.
    if $\{s_{j+1} = \infty\}$,$\tilde{v}_{j+1} = A\tilde{V}_m^j e_j$; **else**
    $\tilde{v}_{j+1} = (s_{j+1} I_n - A)^{-1} \tilde{V}_m^j e_j$; **end**
    $\tilde{v}_{j+1} = \tilde{v}_{j+1} - \tilde{V}_m^j (\tilde{V}_m^j)' \tilde{v}_{j+1}$;
    if $\|\tilde{v}_{j+1}\|$ or $\|\tilde{w}_{j+1}\| < \varepsilon$, Stop **end**
    $v_{j+1} = \tilde{v}_{j+1} / \|\tilde{v}_{j+1}\|$; $\tilde{V}_m^{j+1} = [\tilde{V}_m^j \; v_{j+1}]$;
  end
end
Outputs: $\tilde{V}_m, \tilde{W}_m$

VII. Numerical resutls

In this section the evaluation and comparison of the restart methods suggested in this paper is obtained by applying model reduction on the stable model `kookos180` of order $n = 180$ arising in discretization of single-stage suspension crystalliser [3] which is used as a benchmark in [15]. In order to verify that the restart methods for the modified Arnoldi algorithm perform well, the initial set of interpolation points

$\mathbf{S}_m$ are carefully computed by applying adaptive model reduction on the original systems as described in Section III.

To compare the performance of the restart scheme using the SSM, OSM and EOSM methods, Algorithm 3 was implemented in real arithmetics. To measure the accuracy of the approximations, in each case the relative $\mathscr{H}^\infty$ error $\mathscr{E}_\infty \triangleq \|G - G_m\|_\infty / \|G\|_\infty$, was computed for a number of restarts and the results are shown below at the top of Figure 1. The desired order of the approximation is $m = 12$ and the order of V_r on every restart iteration was set to $r = m-2$ (i.e., on every restart 2 interpolation points were updated). For larger systems the value of r could be increased accordingly.

As expected, the relative error tends to decrease as the number of restarts increases. The SSM methods is expected to perform well while the older and less accurate interpolation points are removed. However after a number of restarts the interpolation points tend to be accurate estimates. Therefore it is suggested that for improved performance while keeping the computational cost low, one could begin restarting the adaptive algorithm using the SSM method and after a number of restarts switch to EOSM or OSM methods. The results are shown for the same systems in Figure 1 on the bottom, where the algorithm switches to OSM and EOSM after 5 restarts.

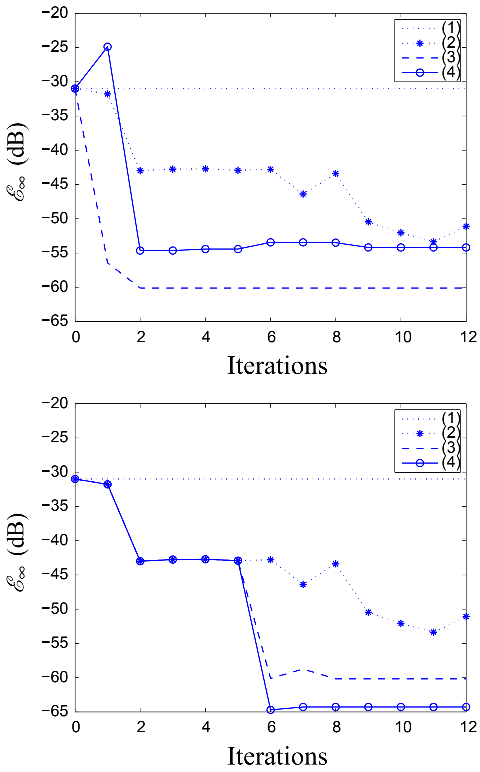

Fig. 1. Error plots for the restart schemes; **Top** : Without restarts (1), with SSM (2), with EOSM (3), with OSM (4). **Bottom** :Without restarts (1), with SSM (2), with SSM-EOSM (3), with SSM-OSM (4)

Remark on convergence: A convergence analysis is out of the scope of this paper. However observing the results in Figure 1, it can be deduced that the schemes involving OSM and EOSM tend to converge after a number of restarts and they generally provide better approximations.

VIII. Conclusions

In this paper we have given an analysis for restarting a modified rational Arnoldi algorithm, while preserving the Arnoldi-like equations derived in [4]. The accuracy of the reduced order system after restarts is improved, while the order of the approximation remains fixed. This is achieved by efficiently updating the interpolation points at which the reduced order model matches the moments of the original system. The new interpolation points are computed adaptively based on simple residual error expressions.

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