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Second order $\mathcal{H}_2$ optimal approximation of linear dynamical systems

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Abstract: In this paper we consider the $\mathcal{H}_2$ optimal model reduction problem which has important applications in system approximation and has received considerable attention in the literature. An important link between special cases of this problem and rational interpolation using Krylov subspace projection methods has been recently established. We use this link to derive a solution of the second order optimal $\mathcal{H}_2$ approximation problem that involves the computation of all simultaneous solutions to two bivariate polynomials. We show that this is equivalent to the simultaneous solution of two 2-dimensional eigenvalue problems. To illustrate our procedure we give a few numerical examples.

1. INTRODUCTION

Model reduction is of significant importance in many areas of science and engineering, involving computations of large-scale systems, such as real-time control, model parameter estimation or weather prediction just to mention a few. Some motivating examples can be found in Antoulas [2005]. To formulate the model reduction problem, let

$$G(s) = C(sI_n - A)^{-1}B, \tag{1}$$

denote the transfer function of a stable, single input single output (SISO) linear time invariant dynamical system, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C^T \in \mathbb{R}^n$ are the system matrices and $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. It is assumed that the dimensionality n of the system is large and the system matrices are sparse. For notation simplicity, in the following we do not always use a subindex on the identity matrix, as its dimension can be infered from the context.

In the model reduction framework one seeks to find a reduced order model of (1) such that it accurately approximates the original model and preserves any desirable properties. The transfer function of the reduced model is

$$G_m(s) = C_m(sI - A_m)^{-1}B_m, \tag{2}$$

where $A_m \in \mathbb{R}^{m\times m}$, $B_m \in \mathbb{R}^m$ and $C_m^T \in \mathbb{R}^m$ are reduced order matrices. The reduced order dimensionality is usually $m \ll n$ which allows efficient computations.

There exist many conventional reduction techniques some of which preserve desirable properties of the original model, and some of which provide global error bounds. Some well known conventional methods are the optimal Hankel norm, balanced and modal truncation methods (Antoulas [2005], Zhou [1995]). A number of conventional-variant techniques utilize iterative solvers for linear matrix equations and parallel programming techniques and are applicable for the reduction of large-scale systems (e.g., Benner et al. [2005], Baur [2008]).

In this paper we are interested in interpolatory model reduction techniques of large-scale linear dynamical systems based on Krylov subspace projection techniques (Antoulas et al. [2010], Gallivan et al. [1996], Grimme [1997], Ruhe [1994a,b]). These methods are very attractive for the reduction of large-scale models due to their numerical stability and computational efficiency.

The key challenge addressed in this paper is the identification of first and second order global $\mathcal{H}_2$ optimal interpolating approximations. The problem of identification of global $\mathcal{H}_2$ optimal approximations, is still an open problem and has received considerable attention in the literature, e.g., Wilson [1970], Hyland and Bernstein [1985], Baratchart et al. [1991], Spanos et al. [1992], Fulcheri and Olivi [1998], Yan and Lam [July 1999] and more recenlty Gugercin et al. [2008]. Most existing approaches derive first order necessary conditions for local optimality. The resulting reduced order systems are not necessarily the global minimizers but often they are effective reduced order approximations.

In Section 2 we formulate the $\mathcal{H}_2$ optimal reduction problem and we discuss its link between the rational interpolation problem. The solution to the optimal $\mathcal{H}_2$ reduction problem is described in Section 3 for SISO systems and an exact expression of the $\mathcal{H}_2$ norm of the error of a general m-th order approximation is derived. In Section 4 we derive a solution for optimal second order approximations and we illustrate the difficulties for the solution to the optimal problem for higher order approximations. Our numerical results are given in Section 5. Finally, we provide our conclusions.

2. FORMULATION

2.1 $\mathcal{H}_2$ optimal model reduction problem

The $\mathcal{H}_2$ optimal model reduction problem is defined as the problem of identifying a stable reduced order system G_m, which is the best approximation of (1) in terms of the $\mathcal{H}_2$-norm. I.e.,

$$G_m = \arg\min_{\hat{G}_m} \|G - \hat{G}_m\|_{\mathcal{H}_2} \tag{3}$$

subject to the constraint that $\hat{G}_m$ is an m-dimensional stable model in the form of (2). For a stable and strictly proper transfer function G, the $\mathcal{H}_2$ norm is defined as,

$$\|G\|^2_{\mathcal{H}_2} := \frac{1}{2\pi}\int_{-\infty}^{+\infty} \text{trace}[G^*(jw)G(jw)]dw = C\mathcal{P}C^T \tag{4}$$

where $G^*(jw)$ is a shorthand for $G^T(-jw)$ and $\mathcal{P}$ is the controllability gramian satisfying

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0.$$

Recently, Gugercin et al. [2008], recognizing that the global solution of this problem is difficult to obtain, they introduce the idea of local minimizers. A stable reduced order approximation of (1), G_m, is a local minimizer for (3), if, for all $\epsilon > 0$,

$$\|G - G_m\|_{\mathcal{H}_2} \le \|G - \tilde{G}_m^{(\epsilon)}\|_{\mathcal{H}_2} \tag{5}$$

for all stable $\tilde{G}_m^{(\epsilon)}$ of order m, such that $\|G_m - \tilde{G}_m^{(\epsilon)}\|_{\mathcal{H}_2} \le C\epsilon$ for some constant C. They also provide an important link to the problem of model reduction by moment matching using rational interpolation via Krylov subspace projection methods. We discuss these in the following.

2.2 $\mathcal{H}_2$ optimal rational interpolating approximations

Rational interpolating approximations are obtained by constructing reduced order models that interpolate (match) the transfer function and perhaps some of the derivatives of G (i.e., the moments), evaluated at a selected set of interpolation points $S_m = \{s_1, \ldots, s_m\}$ in the complex plain. Efficient moment matching can be obtained via the projection framework, without explicit computation of the moments at the required interpolation points, as described in the lemma below.

Lemma 1. (Grimme [1997].) Let $G(s) = C(sI - A)^{-1}B$ and let $V_m, W_m \in \mathbb{R}^{n\times m}$ assuming that $W_m^T V_m$ is nonsingular. Define $G_m(s) = C_m(sI - A_m)^{-1}B_m$ where $A_m = (W_m^T V_m)^{-1}W_m^T A V_m$, $B_m = (W_m^T V_m)^{-1}W_m^T B$, and $C_m = CV_m$. Suppose $\sigma \in \mathbb{C}$ is not an eigenvalue of either A or A_m. If $(\sigma I - A)^{-1}B \in \text{span}(V_m)$ and $(\bar{\sigma}I - A^T)^{-1}C^T \in \text{span}(W_m)$, where $\bar{\sigma}$ denotes conjugation on σ, then the transfer functions $G_m(\sigma) = G(\sigma)$ and their derivatives (with respect to s) $G'_m(\sigma) = G'(\sigma)$, evaluated at $s = \sigma$. □

The Krylov subspace bases V_m and W_m in Lemma 1 can be computed efficiently using the rational Arnoldi Arnoldi [1951] or Lanzos algotithm Lanczos [1950, 1952].

Remark 1. In the following, we occasionally use $V_m(S_m)$, $W_m(S_m)$, $A_m(S_m)$, $B_m(S_m)$ and $C_m(S_m)$ in place of V_m, W_m, A_m, B_m and C_m to show their dependance on S_m.

The link , giving necessary conditions for local optimality, between the $\mathcal{H}_2$ and rational interpolation reduction problems in the case that $G(s)$ is SISO and the poles of the local minimizer are simple, is given in the following result.

Lemma 2. (Gugercin et al. [2008].) Given a stable SISO system $G(s) = C(sI - A)^{-1}B$, let $G_m(s) = C_m(sI_m - A_m)^{-1}B_m$ be a local minimizer (in the sense defined in (5)) of dimension m for the optimal $\mathcal{H}_2$ model reduction problem (3) and suppose that $G_m(s)$ has simple poles at s_i, $i = 1, \ldots, m$. Then $G_m(s)$ interpolates both $G(s)$ and its first derivative at $-s_i$, $i = 1, \ldots, m$:

$$G_m(-s_i) = G(-s_i),\ G'_m(-s_i) = G'(-s_i), \quad i = 1, \ldots, m. \tag{6}$$

□

This link suggests that if we can find all possible sets of interpolating points $S_m = \{s_1, \ldots, s_m\} \subset \mathbb{C}^+$ such that

$$\lambda\left(-A_m(S_m)\right) = S_m, \tag{7}$$

where A_m is defined in Lemma 1 and we use the notation in Remark 1 and $\lambda(\cdot)$ denotes the spectrum, then the corresponding $G_m(s)$'s will include all the local minimizers for $G(s)$. Thus the problem of finding local minimizers reduces to the problem of finding the fixed points of (7). Several algorithms, based on the iteration

$$S_m^{(i+1)} = \lambda(-A_m(S_m^{(i)}))$$

are given in Gugercin et al. [2008], together with some rules for selecting $S_m^{(0)}$, i.e., the set of interpolation points for the initial iteration.

3. $\mathcal{H}_2$ NORM AND MODEL REDUCTION

In this section we describe the solution of the optimal $\mathcal{H}_2$ model reduction problem for SISO systems and derive an expression of the $\mathcal{H}_2$ norm of the approximation error for general m^{th} order approximations. The next theorem gives an interesting result concerning the $\mathcal{H}_2$ norm of the approximation error for all approximations satisfying (7).

Theorem 1. Let $S_m = \{s_1, \ldots, s_m\} \subset \mathbb{C}^+$ be a fixed point of (7) and assume, without loss of generality, that $W_m^T V_m = I_m$. Let $G_m(s) = C_m(sI_m - A_m)B_m$, where $A_m = W_m^T A V_m$, $B_m = W_m^T B$ and $C_m = CV_m$, be the corresponding interpolating function given by Lemma 1, so that $G(s_i) = G_m(s_i)$, $G'(s_i) = G'_m(s_i)$, $i = 1, \ldots, m$ and $\lambda(A_m) = -S_m$. Then

$$\|G - G_m\|^2_{\mathcal{H}_2} = \|G\|^2_{\mathcal{H}_2} - \|G_m\|^2_{\mathcal{H}_2} \tag{8}$$

Proof: It follows from Frangos and Jaimoukha [2008] that the Arnoldi-like equations

$$AV_m = V_m A_m + v_{m+1}C_{V_m},\ B = V_m B_m + v_{m+1}D_{V_m} \tag{9}$$
$$W_m^T A = A_m W_m^T + B_{W_m} w_{m+1}^T,\ C = C_m W_m^T + D_{W_m} w_{m+1}^T$$

can be constructed, where all the terms are defined in Frangos and Jaimoukha [2008]. Using these equations, the error system $E_m(s) := G(s) - G_m(s)$ can be written as

$$\begin{aligned} E_m(s) &= C(sI - A)^{-1}B - C_m(sI - A_m)^{-1}B_m \\ &= C(sI - A)^{-1}v_{m+1}R_m(s) \end{aligned}$$

where

$$R_m(s) = D_{V_m} + C_{V_m}(sI - A_m)^{-1}B_m \tag{10}$$

is the residual error. Note that the interpolation points are the zeros of $R_m(s)$, Frangos and Jaimoukha [2008] and, since the poles of $R_m(s)$ are the eigenvalues of A_m it follows that $D_{V_m}^{-1}R_m(s)$ is allpass. Thus if P_m is the controllability Grammian, then

$$A_mP_m+P_mA_m^T+B_mB_m^T=0, \quad C_{V_m}P_m+D_{V_m}B_m^T=0 \tag{11}$$

are the allpass equations for $R_m(s)$ Zhou et al. [1996]. Let

$$P_E = \begin{bmatrix} P & X \\ X^T & P_m \end{bmatrix} \tag{12}$$

be the controllability Grammian for

$$E_m(s) \overset{s}{=} \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A_m & B_m \\ \hline C & -C_m & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_E & B_E \\ \hline C_E & D_E \end{array}\right] \tag{13}$$

so that

$$\begin{bmatrix} AP+PA^T+BB^T & AX+XA_m^T+BB_m^T \\ \star & A_mP_m+P_mA_m^T+B_mB_m^T \end{bmatrix}=0. \tag{14}$$

Then a manipulation using (9), (11) and (14) gives

$$A(XP_m^{-1}-V_m) = (XP_m^{-1}-V_m)(A_m-B_mD_{V_m}^{-1}C_{V_m}) \tag{15}$$

Since the interpolation points are the eigenvalues of $A_m - B_mD_{V_m}^{-1}C_{V_m}$ (i.e. the zeros of $R_m(s)$) and since it is assumed that the interpolation points are not eigenvalues of A, it follows that $V_m = XP_m^{-1}$. Thus

$$CX = CV_mP_m = C_mP_m. \tag{16}$$

Now using the expression for the $\mathcal{H}_2$ norm in (4)

$$\begin{aligned} \|G - G_m\|_{\mathcal{H}_2}^2 &= C_EP_EC_E^T \\ &= CPC^T - C_mX^TC^T - CXC_m^T + C_mP_mC_m^T \\ &= CPC^T - C_mP_mC_m^T \\ &= \|G\|_{\mathcal{H}_2}^2 - \|G_m\|_{\mathcal{H}_2}^2 \end{aligned}$$

where the third equation follows from (16) and this proves the theorem. □

The next theorem gives an explicit solution of the $\mathcal{H}_2$ norm model reduction problem in the case that $m = 1$ and uses Theorem 1.

Theorem 2. (Ahmad et al. [2010]) Let $G(s) = C(sI_n - A)^{-1}B$ be a stable SISO system and let $m = 1$. Then all fixed points of (7) are given by the zeros of the transfer function

$$\tilde{H}(s) = \tilde{C}(sI_{2n} - \tilde{A})^{-1}\tilde{B} \overset{s}{=} \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right] := \left[\begin{array}{cc|c} A & A & B \\ 0 & A & 2B \\ \hline C & 0 & 0 \end{array}\right] \tag{17}$$

or, equivalently, by the eigenvalues of the matrix pencil

$$\begin{bmatrix} \tilde{A} - \lambda I_{2n} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}.$$

Let $s_1, \ldots, s_f$ be the positive zeros of $\tilde{H}$ and let $G_1(s), \ldots, G_f(s)$ be the corresponding interpolating transfer functions given by Lemma 1, so that

$$G_i(s_i)=G(s_i), \; G_i^{'}(s_i)=G^{'}(s_i), \; i=1,\ldots,f. \tag{18}$$

Then the global solution to the $\mathcal{H}_2$ model reduction problem (3) is the $G_i(s)$'s with the maximum $\mathcal{H}_2$ norm.

4. SECOND ORDER APPROXIMATION

Although the solution of the $\mathcal{H}_2$ norm model reduction problem in the case $m = 1$ was given explicitly in the previous section, the solution in the general case is much more difficult. The following result derives a solution to the $m = 2$ problem that involves the evaluation of all simultaneous solutions of two equations in two unknowns.

Theorem 3. Let all variables be as defined before and let $m = 2$. For $s_1, \; s_2 \in \mathbb{C}^+$ such that $S_m := \{s_1, s_2\}$ is closed under conjugation define

$$\begin{aligned} V_m(s_1,s_2) &= \left[(s_1I-A)^{-1}B \quad (s_1I-A)^{-1}(s_2I-A)^{-1}B\right] \\ W_m(s_1,s_2) &= \left[(s_1I-A)^{-T}C^T \quad (s_1I-A)^{-T}(s_2I-A)^{-T}C^T\right] \\ A_m(s_1,s_2) &= \left(W_m^T(s_1,s_2)V_m(s_1,s_2)\right)^{-1}W_m^T(s_1,s_2)AV_m(s_1,s_2) \\ f(s_1,s_2) &= \det\left(W_m^T(s_1,s_2)(s_1I+A)V_m(s_1,s_2)\right). \end{aligned}$$

Then all $s_1, \; s_2 \in \mathbb{C}^+$ such that $\{s_1, s_2\} = -\lambda\left(A_m(s_1, s_2)\right)$ are given by the simultaneous solutions of the two equations

$$f(s_1, s_2) = 0, \qquad f(s_2, s_1) = 0.$$

Proof: s_1 is an eigenvalue of $-A_m(s_1, s_2)$ if there exists a nonzero $x_m \in \mathbb{C}^m$ such that $(A_m(s_1, s_2) + s_1I)\, x_m = 0$. A simple manipulation shows that this is equivalent to $f(s_1, s_2) = 0$. Note that since $\lambda\left(A_m(s_1, s_2)\right) = \lambda\left(A_m(s_2, s_1)\right)$ (moment matching is independent of the order it is carried out), it follows that s_2 is an eigenvalue of $-A_m(s_1, s_2)$ if there exists a nonzero $x_m \in \mathbb{C}^m$ such that $(A_m(s_2, s_1) + s_2I)\, x_m = 0$. A simple manipulation shows that this is equivalent to $f(s_2, s_1) = 0$. □

The following result shows that the solution to the $m = 2$ problem reduces to the evaluation of all simultaneous solutions of two bivariate polynomial equations. The proof involves standard linear algebra and is omitted.

Theorem 4. Let all variables be as defined before and let $m = 2$. Define

$$\begin{aligned} \mathcal{A}_{11} &= \begin{bmatrix} A & 2A \\ 0 & A \end{bmatrix} & \mathcal{A}_{12} &= \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix} & \mathcal{B}_1 &= \begin{bmatrix} B & 0 \\ B & 0 \end{bmatrix} \\ \mathcal{A}_{21} &= \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} & \mathcal{A}_{22} &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} & \mathcal{B}_2 &= \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \\ \mathcal{C}_1 &= \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} & \mathcal{C}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} & \mathcal{D} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$p(s_1, s_2) = \det\left(\begin{bmatrix} \mathcal{A}_{11} - s_1I & \mathcal{A}_{12} & \mathcal{B}_1 \\ \mathcal{A}_{21} & \mathcal{A}_{22} - s_2I & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{D} \end{bmatrix}\right).$$

Then all $s_1, \; s_2 \in \mathbb{C}^+$ such that $\{s_1, s_2\} = -\lambda\left(A_m(s_1, s_2)\right)$ are given by the simultaneous solutions of the two bivariate polynomial equations

$$p(s_1, s_2) = 0, \qquad p(s_2, s_1) = 0 \tag{19}$$

□

Note that the solution is given in terms of the simultaneous solutions of two 2-dimensional matrix pencil problems. Under a mild assumption, the next result gives the solution in terms of the simultaneous solutions of two 2-dimensional matrix eigenvalue problems.

Theorem 5. Let all variables be as defined before and let $m = 2$. Assume that $p(0,0) \neq 0$ so that the pair $s_1 = s_2 = 0$ is not a solution to (19). Let

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}, \quad \mathcal{B} = [\, \mathcal{B}_1 \ \mathcal{B}_2 \,], \quad \mathcal{C} = \begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{bmatrix}$$

Assume that

$$\mathcal{C}\mathcal{A}^{-1}\mathcal{B} = \begin{bmatrix} -CA^{-1}B & CA^{-2}B \\ CA^{-2}B & -CA^{-3}B \end{bmatrix}$$

is nonsingular (which is equivalent to $f(0,0) \neq 0$), let

$$\hat{\mathcal{A}} = \mathcal{A}^{-1} - \mathcal{A}^{-1}\mathcal{B}\left(\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)^{-1}\mathcal{C}\mathcal{A}^{-1}$$

and partition $\hat{\mathcal{A}}$ compatibly with $\mathcal{A}$ as

$$\hat{\mathcal{A}} = \begin{bmatrix} \hat{\mathcal{A}}_{11} & \hat{\mathcal{A}}_{12} \\ \hat{\mathcal{A}}_{21} & \hat{\mathcal{A}}_{22} \end{bmatrix}.$$

Let $\hat{s}_1$ and $\hat{s}_2$ denote s_1^{-1} and s_2^{-1}, respectively, and define

$$\hat{p}(\hat{s}_1, \hat{s}_2) = \det\left(\begin{bmatrix} \hat{\mathcal{A}}_{11} - \hat{s}_1 I & \hat{\mathcal{A}}_{12} \\ \hat{\mathcal{A}}_{21} & \hat{\mathcal{A}}_{22} - \hat{s}_2 I \end{bmatrix} \right).$$

Then all $s_1,\ s_2 \in \mathbb{C}^+$ such that $\{s_1, s_2\} = -\lambda\left(A_m(s_1, s_2)\right)$ are given by $s_1 = \hat{s}_1^{-1}$ and $s_2 = \hat{s}_2^{-1}$ where $\{\hat{s}_1, \hat{s}_2\}$ are simultaneous solutions of the two bivariate polynomial equations

$$\hat{p}(\hat{s}_1, \hat{s}_2) = 0, \qquad \hat{p}(\hat{s}_2, \hat{s}_1) = 0 \tag{20}$$

□

5. NUMERICAL RESULTS

To illustrate the procedure described in section 4, we give some numerical examples. Consider the following (randomly generated) transfer functions,

$$F_1(s) = \frac{-2.9239s^3 - 39.5525s^2 - 97.5270s - 147.1508}{s^4 + 11.9584s^3 + 43.9119s^2 + 73.6759s + 44.3821},$$
$$F_2(s) = \frac{-1.2805s^3 - 6.2266s^2 - 12.8095s - 9.3373}{s^4 + 3.1855s^3 + 8.9263s^2 + 12.2936s + 3.1987},$$
$$F_3(s) = \frac{-1.3369s^3 - 4.8341s^2 - 47.5819s - 42.7285}{s^4 + 17.0728s^3 + 84.9908s^2 + 122.4400s + 59.9309}.$$

For the three systems above, second order optimal approximations are constructed, $\hat{F}_1, \hat{F}_2$ and $\hat{F}_3$, following the approach presented in this work. The choice of fixed points are shown in Table 1. The relative $\mathcal{H}_2$ norm of the approximation error,

$$E_{\hat{F}}(S) := \|\hat{F}(S) - F(S)\|_{\mathcal{H}_2} / \|F(S)\|_{\mathcal{H}_2},$$

is given in Table 2 evaluated at the fixed points.

Table 1. Fixed points for $\hat{F}_1$, $\hat{F}_2$, $\hat{F}_3$

	Fixed Points $S = \{s_1, s_2\}$
$\hat{F}_1$	$S_1 = \{2.4437, 0.8883\}$
	$S_2 = \{42.8733, 0.9891\}$
$\hat{F}_2$	$S_1 = \{1.2052, 0.2030\}$
	$S_2 = \{6.3626, 1.1693\}$
$\hat{F}_3$	$S_1 = \{0.8261+0.6577i, 0.8261-0.6577i\}$
	$S_2 = \{39.2800, 0.7051\}$

Table 2. Relative error for $\hat{F}_1$, $\hat{F}_2$, $\hat{F}_3$

	Local Minimizers $E_{\hat{F}}(S_1)$	Local Minimizers $E_{\hat{F}}(S_2)$	Global Minimizer
$\hat{F}_1$	0.0546	0.0563	0.0546
$\hat{F}_2$	0.3271	0.3370	0.3271
$\hat{F}_3$	0.2998	0.2676	0.2676

Next, we repeat our approach to obtain second order optimal approximations for some linear systems used in the literature, given below,

- F_4 in Hyland and Bernstein [1985], given in state space representation $F_4(s) \overset{s}{=} (A, B, C, 0)$, with
$$A = \begin{bmatrix} 0 & 0 & 0 & -150 \\ 1 & 0 & 0 & -245 \\ 0 & 1 & 0 & -113 \\ 0 & 0 & 1 & -19 \end{bmatrix}, \ B = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ C^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$
- F_5 in Spanos et al. [1992], given by,
$$F_5(s) = \frac{s^2 + 15s + 50}{s^4 + 5s^3 + 33s^2 + 79s + 50},$$
- F_6 in Halevi [1992], given by,
$$F_6(s) = \frac{41s^2 + 50s + 140}{(s^2 + s + 1)(s^2 + 10s + 100)}.$$

Table 3. Fixed Points for $\hat{F}_4, \hat{F}_5, \hat{F}_6$

	Fixed Points $S = \{s_1,\ s_2\}$
$\hat{F}_4$	$S_1 = \{2.5113,\ 1.0990\}$
$\hat{F}_5$	$S_1 = \{4.1935,\ 1.1539\}$
$\hat{F}_6$	$S_1 = \{0.6115, 0.3231\}$
	$S_2 = \{4.9524, 0.5837\}$
	$S_3 = \{73.6648, 0.8971\}$
	$S_4 = \{27.2067, 3.8413\}$
	$S_5 = \{8.8684 + 5.2588i, 8.8684 - 5.2588i\}$

Table 4. Relative error for $\hat{F}_4, \hat{F}_5, \hat{F}_6$

	Local Minimizers $E_{\hat{F}}(S_1)$	$E_{\hat{F}}(S_2)$	$E_{\hat{F}}(S_3)$	$E_{\hat{F}}(S_4)$	$E_{\hat{F}}(S_5)$	Global Minimizer
$\hat{F}_4$	0.0393	-	-	-	-	0.0393
$\hat{F}_5$	0.2443	-	-	-	-	0.2443
$\hat{F}_6$	0.5434	0.5078	0.5336	0.5883	0.5840	0.5078

Similarly, we follow the approach presented in this work to construct second order optimal approximations $\hat{F}_4, \hat{F}_5$ and $\hat{F}_6$. The fixed points and their corresponding relative errors are shown in Table 3 and Table 4 respectively. Note that, there is only one fixed point for models $\hat{F}_4$ and $\hat{F}_5$. For comparison to the approach presented in this work, we mention that the iterative rational Krylov method $S^{(i+1)} = \lambda(-A_m(S^{(i)}))$ converges to the fixed points for any initial choice of the starting interpolation point set. However, in the case of $\hat{F}_6$, where there are five fixed points, the convergence of the iterative method to the optimal points depends on the choice of the initial interpolation points set.

6. CONCLUSION

In this paper we derive the solution to the optimal $\mathcal{H}_2$ model reduction problem for SISO systems, for second order approximations via rational interpolation. We show that the problem reduces to the evaluation of all simultaneous solutions of two bivariate polynomial equations and is equivalent to the simultaneous solution of two 2-dimensional eigenvalue problem. Efficient computation of multi-dimensional eigenvalue problem for higher order $\mathcal{H}_2$ optimal model reduction problem is the subject of our ongoing research.

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